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THE EVOLUTION OF CONJECTURES

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Abstract.

In this paper we model the evolution of conjectures in an economy consisting of a continuum of firms which meet in duopolies according to a random matching framework. The duopoly game is modelled by the conjectural variation model, where the firms belief (conjecture) about the other firm's behaviour determines its own behaviour. An evolutionary process occurs, by which conjectures that lead (on average) to higher profits become more common. This is modelled by replicator dynamics, and can be seen as occurring due to a process of imitation, propagation, and natural selection. In the context of homogeneous good Cournot duopoly, the conjectures resulting from the noiseless replicator dynamics are the consistent conjectures. We also obtain a variety of analytic and simulation results for noisy replicator dynamics, and equilibrium selection.

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In this paper we explore the evolution of beliefs in an economy in which boundedly rational firms are randomly matched in pairs each period to play a duopoly. The firms have beliefs about how their opponents play the duopoly game. The population of firms is characterised by a distribution of firm types: a firm type represents a particular *belief* that the firm might have. When we look at the population (economy) as a whole, some types of belief will (on average) be more profitable, and others less profitable. In this paper, we argue that beliefs (firm types) that are more profitable will become more common; beliefs that are less profitable will become less common. Hence we envisage a process of Darwinistic social evolution driving the distribution of beliefs of firms in the economy. It should be noted that this is an *economic* or *strategic* explanation of the beliefs of firms. This contrasts with much of economic theory which adopts an *epistemic* view, where rational firms adjust beliefs in a way which is in some sense statistically optimal (e.g. using Bayes rule under common knowledge).

The nature of the social evolution can be viewed in several ways. It can be a process of *imitation*: less successful firms imitate the more successful. It can be a process of *propagation*, in that the best practice of successful firms is spread by some mechanism: the successful firms diversify (multiply), the managers of successful firms move around and bring the ideas of the successful firm to the less successful firm types. It can also be a method of *selection*: the least successful firms are more likely to go bankrupt than the more successful. In this paper, we do not attempt to model this process in detail. Rather, we assume that there is indeed some Darwinian mechanism at work, and in particular we model this by replicator dynamics[1].

We model firms' beliefs within the conjectural variation Cournot duopoly model with a homogeneous good. Each firm has a (linear) decision rule, which gives its output as a function of the output of the other firm. The decision rule of firm i is related to its conjecture about the slope of the other firm j's decision rule: firm i maximises its own profit given its conjecture about

[1]Replicator dynamics and/or similar evolutionary algorithms have been the focus of recent literature on game theory. See, among the others, Binmore and Samuelson (1992, 1995), Binmore *et al.* (1994), Linster (1992, 1994), Nachbar (1992), Kandori *et al.* (1993).

the slope of the other firm's output. We can classify firms in the economy according to the value of this conjecture; a firm *type* is a particular value of conjecture ϕ. The range of firm types we consider is from ϕ=-1 (the Bertrand conjecture), through ϕ=0 (the Cournot or Nash conjecture) to ϕ=+1 (the joint profit maximising or perfectly cooperative firm). The proportion of firms that are of a particular type i is Z_i. Clearly, the average profit of a particular firm type will depend on the distribution of firm types in the population. Changes in the population proportions are driven by the profits of the firm type relative to the average population profits: the profits of each firm type respond to population dynamics. We consider the case of "noisy" replicator dynamics, so that there is some randomness in the process. This process may converge to a steady state where population proportions are constant.

The results of this paper are very clear cut, and partly rely on analysis and partly on simulation. First, we show that in the case of linear demand and quadratic costs, there is a unique attractor for the replicator dynamics of this model when there is no noise: the conjectures of the firms will converge onto the *consistent* conjectures[2]. This is an interesting result: the original literature on consistent or rational conjectures saw the justification as being in terms of epistemic rationality: rational firms ought to be correct. Here, however, we can see an alternative explanation: a population of boundedly rational firms can evolve to the consistent conjectures. In the case of no costs of production, the consistent conjecture is the Bertrand conjecture ϕ=-1. This is a "dominated strategy", and not evolutionary stable. We analyse the case where we have a finite set of firm types (the set is generated by a grid of granularity δ on the interval [-1, 1]). There is a set of attractors consisting of firm types -1 to some $\phi^*>-1$. These are Pareto ranked attractors. We find that from a wide range of simulations the dynamics converge on the Pareto Dominant equilibrium.

[2]By consistent, we mean that both firms conjectures are equal to the slope of the other firm's reaction function. Since the decision rules are linear, this is equivalent to Bresnahan's definition (1981). Other early contributions include Hahn (1977,1978), Perry (1980), Kamien and Schwartz (1983), Ulph (1983), Boyer and Moreaux (1983).

With a little bit of "noise", the replicator dynamics lead to a distribution which is massed around the point attractor of the noiseless dynamics. However, as the level of noise increases, we find that distribution becomes much flatter.

We believe that the methodology of this paper highlights an important idea. One of the explanations of types of behaviour and of the underlying beliefs is that they lead to higher payoffs and hence become more common. This is not a form of explanation based on some fundamental notion that firms (or economic agents) are "rational" in any fundamental sense, but merely that they are "intelligent" and boundedly rational, and that there may also be some process of natural selection rewarding the successful. In the context of conjectural variation duopoly, we come up with the surprising result that the evolutionary process provides a possible justification for consistent conjectures based on bounded rationality. The main objection to the notion of consistent conjectures in the past was that it did not fit in with the notions of rationality and common knowledge of game theory[3]. The results of this paper suggest that more recent developments in evolutionary game theory may lead to a reappraisal of the notion of consistent conjectures.

1: The basic duopoly model: conjectural variation duopoly

We are considering an economy populated by firms matched in pairs, each pair playing a duopoly. Let us first consider the constituent duopoly game, which in this paper is the simplest model of conjectural variation duopoly. Firms choose output levels which are produced with marginal cost $TC_i = \frac{c}{2}q_i^2$ i=a,b. The market price is assumed to be a linear function of the two outputs $p = 1 - q_a - q_b$. Firm a's payoff function is given by

$$\Pi_a(q_a, q_b) = q_a\left(1 - q_a - q_b - \frac{c}{2}q_a\right) \qquad [1]$$

[3]In Tirole words "We conclude that this methodology (the conjectural variation approach) is not theoretically satisfactory, as it does not subject itself to the discipline imposed by game theory" (1988, p.244). Other criticism can be found in Makowski (1987), Shapiro (1989), Lindh (1992).

Each firm has a conjecture about how the other firm will respond to it. We call this the conjectural variation parameter ϕ (CVP), where:

$$\phi_a \equiv \frac{dq_b}{dq_a} \qquad [2]$$

The maximisation of profits [1], given the CVP [2] yields the first order condition:

$$\frac{d\Pi_a}{dq_a} = 1 - \left[(2+\phi_a + c)q_a + q_b\right] = 0 \qquad [3]$$

Equation [3] defines the "reaction function" of firm a given its belief ϕ, which we will describe here as a "decision rule". There is a population of firm types, a firm type is defined as a conjecture, which is represented by the corresponding decision rule. A decision rule for firm a (S_a) is a function mapping the output of the other firm and its own conjecture ϕ onto a choice of output:

$$S_a \equiv f_a : \{q_b, \phi_a\} \Rightarrow [0,1]$$

A convenient way to represent a firm type is by its explicit decision rule, derived from [3], which is composed by an intercept term and a slope term (dropping the subscript):

$$h_0 = \frac{1}{2+\phi+c} \qquad \text{(intercept)} \qquad [4a]$$

$$h_1 = -\frac{1}{2+\phi+c} \qquad \text{(slope)} \qquad [4b]$$

Both the intercept and the slope depend on ϕ which, it is useful to recall, is a measure of the expected competitiveness of the rival. Given a pair of decision rules, the equilibrium output pair is given by the point of intersection. Assuming c>0 we can define the closure of the set of output pairs yielding strictly positive profits[4] for firms a and b:

$$A_a : A_a \equiv \left\{ (q_a, q_b) \in [0,1]^2 : q_a \le (1-q_b)\left(\frac{2}{2+c}\right)\right\} \qquad [5]$$

[4]If c>0, then A_i is the set of points with non-negative profits: if c=0, then since profits are never below zero, we define A_i as the closure of the set of points yielding strictly positive profits.

and similarly for firm b; we define next the set of output s yielding positive profits to both firms as $A = A_a \cup A_b$ and depict A_a, in fig.1(a) and A in fig.1(b).

[fig.1 here]

The set $1 - q_a - q_b = cq_a$ corresponds to the Walrasian set, where price is equal to marginal costs. The Cournot-Nash outcome occurs at point C where both firms produce 1/(3+c) and the price is (1+c)/(3+c). The monopoly point for firm a (b) is the point M_a (M_b) where the firms produce the monopoly output 1/(2+c) and the price is (1+c)/(2+c). The points S_a and S_b represent the Stackelberg points for firm a and b respectively. Given that we restrict ourselves to $\phi \in [-1,1]$, the set of all possible equilibria is represented by the shaded area in fig.1(b)

The "genetic variability" among firm types reduces to the different values assumed by the CVPs. We will restrict our analysis to the set $\phi \in [-1,1]$. The lower bound on the CVP ensures that $q_a + q_b \le 1$ when c=0. The imposition of an upper bound is less obvious; note that for $\phi_a = \phi_b = 1$ the equilibrium will be the Joint Profit Maximising (JPM) outcome at point J in fig.1 corresponding to the symmetric joint profit maximisation. For this reason, as a first approximation, we impose the upper bound on ϕ. The resulting range of firm types runs from the *Walrasian* firm producing output up to the point where price equals marginal costs to the *cooperative* firm. We can infact consider various standard types of decision rules, represented in Table 1:

		C=0		C=0.5		C=1	
Firm Type	ϕ	Intercept	Slope	Intercept	Slope	Intercept	Slope
Cournot Firm	0	0.5	-0.5	0.4	-0.4	1/3	-1/3
Walrasian Firm	-1	1	-1	2/3	-2/3	0.5	-0.5
Cooperative(JPM)	1	1/3	1	2/7	-2/7	1/4	-1/4
Stackelberg (SLF)	-1/(2+c)	2/3	-2/3	10/11	-10/11	3/8	-3/8

Table 1 Some standard decision rules.

The Cournot firm believes that the output of its rival will remain unchanged following a variation in its own output; the Walrasian firm is the most competitive, it believes that the opponent will react to its output changes with an equal and opposite change in output so as to maintain a constant market price. The Joint Profit Maximiser firm (JPM), or cooperative firm, expects the opponent to match its own behaviour, i.e. following a reduction in output, it will expect an equal reduction in the output of the opponent. Finally the Stackelberg Leader Firm (SLF) expects its rival to behave like a Cournot firm and hence its CVP is equal to -1/(2+c), the slope of Cournot firm's decision rule (which depends on c).

It is easy to verify that for pairs of CVP's between [-1,1] a stage game equilibrium always exists and is stable when c>0. When c=0, the stage game is stable unless both firms have conjectures equal to minus unity[5]. This allow us to compute the equilibrium pair $\{q_a, q_b\}$ and the related payoffs as a function of each firm's CVP. In particular, from [4]

$$q_a^{*}(\phi_a, \phi_b) = \frac{1+\phi_b+c}{3+4c+c^2+(2+c)(\phi_a+\phi_b)+\phi_a\phi_b} \qquad [6]$$

Substituting [6] into [1] yields:

$$\Pi_a^{*}(\phi_a, \phi_b) = \frac{(1+\phi_b+c)^2(2+c+2\phi_a)}{2(3+4c+c^2+(2+c)(\phi_a+\phi_b)+\phi_b\phi_a)^2} \qquad [7]$$

Expressions [6] and [7] give the equilibrium output and profits respectively. In terms of fig.1(b), the equilibrium output pairs will be restricted to the shaded compact convex subset of A. If both firms have a CVP=1, the stage game equilibrium will occur at J; When both firms have CVP=-1

[5]In particular, we can represent the decision rules as defined by a dynamic system of the form $\underline{q}_t = \underline{h} + H\underline{q}_{t-1}$ where

$$\underline{h} = [h_{0a}, h_{0b}],\ H = \begin{bmatrix} 0 & h_{1a} \\ h_{1b} & 0 \end{bmatrix}$$

and h_0 and h_1 are defined as in [4]. The equilibrium of this dynamic system is stable if both eigenvalues are real and less than unity in absolute value. The Eigenvalues are the root of $\sqrt{h_{1a}h_{1b}}$. It is easy to verify that this is always the case when CVP are restricted between [-1,1] and c>0. If c=0 with both CVPs equal to -1, we get a positive unit root. In this case the level of profits depends on the initial output. We adopt the convention that in this case, profits are at their equilibrium value of zero.

equilibrium will occur at W, if both firms have Cournot conjectures CVP=0, the equilibrium will occur at C.

2: The Population of firm types and Axelrod Tournaments

The method we adopt consists of three stages. First we determine the set of firm types which will constitute our "population". Second the firms play an Axelrod Tournament, in which each firm type plays each other and we determine the payoff for each possible pairing. The Axelrod Tournament is simply a way of generating the payoff matrix of all possible pairings of firm types. We will run Tournaments for different values of c the cost parameter. Thirdly, given the payoff matrix generated in stage 2, we run an evolutionary algorithm based on the "replicator dynamics".

Generating firm types is an important step in the procedure. The nature of the final equilibrium, if any, will depend crucially on the initial population assumed, since the replicator dynamics we consider does not create new strategies. The algorithm employed to generate firm types is a grid search on the segment [-1,1] with granularity δ. We need only to specify the granularity of the grid, i.e. the distance between two adjacent points on the segment. For all our simulations, we have chosen δ=0.02 which generates 101 types of firms with a CVP including both -1 and 1. We will denote the ordered set of firm types as Φ where:

$$\Phi \equiv \left\{\phi_i : \phi_i = -1 + \delta * i,\ i = 0, \ldots, \frac{2}{\delta}\right\}$$

The set of firm types, then play a hypothetical tournament, which we called an Axelrod Tournament given the similarity with the renowned one (Axelrod, 1984). Each firm type faces each other firm type in a conjectural variation duopoly game and receives a payoff depending on the firm type with which it is matched. Calculating payoffs is very easy. Infact, restricting our attention to the set Φ a closed form solution to the oligopoly game always exists and it is given by [6]. Once the complete tournament is run, a nxn payoff matrix, indicating the payoff received by each firm playing some other, is obtained. More formally, let π_{ij} be the payoff of firm type i playing firm type j. Then the matrix $T \equiv [\pi_{i,j}]$ where i,j=1,2,..,n, denote row and column respectively. The Average Tournament Profit (ATP) of firm i is then given by the average of firm type i's payoff summed over row i:

$$ATP_i = \sum_{j=1}^{n} \pi_{i,j}$$

The average ATP of the tournament is then the average of all individual ATPs.

We have run one tournament with δ=0.02 and n=101 firm types involving 10121 single market games. In Table 2 we present the results for some interesting firm types for the case of c=0:

Firm Type	φ	ATP	Rank (n=101)
Champion	-0.48	0.109551	1
Cournot Firm	0	0.098634	38
WF	-1	0	101
JPM	1	0.072303	94
SLF	-0.5	0.109548	2
Superfirm		0.1123793	

Table 2. Some results of the Axelrod tournament with c=0 and granularity δ=0.02

The champion of the tournament (Champ.), is the firm type earning the highest ATP in a given Tournament. Here, with c=0, the champion has a conjectural variation parameter (φ) equal to -0.48, very close to that of the Stackelberg Leader Firm (SLF) which in turn performs quite well placing itself second. In general, as we would expect, in the single shot confrontation competitive firms, characterised by negative values of φ, perform better. Indeed the *cooperative* Firm (JPM) -with φ=1- earns a very low ATP and ranks at the 94th place. On the other extreme the *Walrasian* Firm (WF), with φ=-1, receives zero profits in every game. Another interesting firm type is the Cournot Firm (φ=0) which comes in at 34th place. In order to put these results in

perspective, a useful comparison can be made with the ATP of a firm which chooses optimally its conjectural variation parameter against each other firm. As in Dixon *et al* (1994) we define this firm as the "Superfirm". In the tournament we assume that each firm type plays always with the same φ; the ratio between each ATP and that of the Superfirm will give a measure of the loss from bounded rationality. The ATP of the Champ is equal to 0.109551 and represents the 97% of the profit earned by the Superfirm which is pretty high given the large variety of firm types considered.

In table 3 we summarise the results of Tournaments for different values of c: we chose the values {0.1, 0.3, 0.5, 1.0}. The first column gives the value of c, the second gives the CVP of the champion firm for each value of c, the third column gives the champion ATP and the fourth the ratio of champion to Superfirm profits for each Tournament.

C	Champion	ATP	%SF
0	-0.48	0.1096	97.6
0.1	-0.48	0.1073	98.3
0.3	-0.44	0-1037	99.3
0.5	-0.4	0.1003	99.6
1	-0.34	0.0926	99.9

Table 3: The Tournament for different values of c and granularity δ=0.02

We can easily represent all of the results of the 5 Tournaments by plotting graph for each value c which relates the CVP to the corresponding ATP.

Fig.2 here

It is worth noting here that as c becomes larger the champion firm becomes *less competitive*, but the CVP is still strictly negative. The ATP of the champion and superfirm both decline, but the ratio of champion to superfirm profits increases (the fact that profits decline reflects the fact that costs are higher for the outputs as c increases).

3: The evolution of conjectures

In the previous stage we generated numerous different firm types and run an Axelrod tournament amongst the firms. The resulting payoff matrix T indicates the equilibrium payoff every possible firm will get when confronted with any other firm in the conjectural variation duopoly game. The next step is therefore to allow for the population of firms to evolve. Firm types which have higher profits (on average) become more common. This can occur through the imitation of more successful firm types by the less successful, or through less successful firm types going bankrupt, or through the propagation of successful firm types through acquisition or diversification. We do not model the process explicitly.

The population consists of al large number of firms whose distribution across firm types is summarised by a state vector $Z \equiv \{z_1, z_2, \ldots, z_n\}$, whose elements z_i represents the distribution of the population on Φ. The vector Z gives then the proportions of each firm type existing in that period. Consider an initial distribution Z^0 on Φ; each period firms are randomly matched and play the conjectural variation duopoly game. The dynamics of the distribution is governed by replicator dynamics which posits that the proportion of firm types in each period evolves according to the individual performance with respect to the average ATP. Firm types which perform better than average in the present period will increase their proportion in the next, while less successful strategies will become less common.

Accordingly we adopt the "noisy" replicator dynamics, as in Gale *et al.* (1995):

$$z_{i,t} = (1-d)\left(z_{i,t-1} + z_{i,t-1}\left[\Pi_{i,t-1} - \overline{\Pi}_{t-1}\right]\right) + d\frac{1}{n} \qquad [8]$$

where $d \in [0,1]$, $\Pi_{i,t}$ is the average payoff of firm type i at time t:

$$\Pi_{i,t} = \sum_{j=1}^{n} z_{j,t}\pi_{i,j} \qquad [9]$$

and $\overline{\Pi}_{t-1}$ is the average payoff of all firms in iteration t:

$$\overline{\Pi}_t = \sum_{i=1}^{n} z_{i,t}\Pi_{i,t} \qquad [10]$$

The parameter d is a noise parameter.

Consider first the case with no noise, d=0. Here, two aspects should be stressed: i) the proportions of firm types evolve according to the absolute difference of profits from the population average; this implies that better (worse) strategies will become more (less) common, ii) the proportion of a strategy in the next iteration depends not only on its own relative performance, but also on its proportion in the present population. This capture the idea that in order to be imitated (or avoided) , strategies must be both successful (bad) and visible. The replicator dynamics may eventually converge to an equilibrium where all surviving firm types earn the same average profit:

$$Z^* \equiv \{z: \Pi_i = \Pi_j > 0\}, \text{ if } z_i, z_j \neq 0 \ \forall\, i, j = 1, \ldots, 101.$$

This equilibrium corresponds to an attractor. Firm types can be considered as *pure strategies* in a game. If it exists, Z^* is then a Nash-equilibrium given the set of strategies that were present in the starting population[6].

With d>0, we introduce noise in the replicator dynamics. In particular, we assume that a fraction d of firms randomly switch type with probability (1/101). This is just one of many possible types of noise, and it could be interpreted as a special type of random mutation that does not create new firm types. In each iteration a fraction d of each firm type is reintroduced, in the resulting equilibrium, if any, all firm type will be present with $z_i \geq d$. As it will become clear later, the introduction of noise in the replicator dynamics performs an important task. Namely, it

[6]On this interpretation and on the issue of the "no creation property", see Nachbar, proposition 1, p.313, (1992).

will make the results more robust by making it independent of the initial conditions[7]. With noise, the criterion for convergence is that the change in population proportions Z is such that the largest change in any z_i is less than 1e-100.

The replicator dynamics two properties which will prove useful in the following analysis:

[FI] *forward invariance*;

[BP] *boundary property*;

[FI] implies that according to [8] strategies are neither created nor, except in the limit destroyed; [BP] states that if $z_{i,T} > 0$ and $z_{j,T} > 0$, and $\Pi_{i,t} > \Pi_{j,t}$ for some t>T, then $lim\, z_{i,t} = 0$ implies that $lim\, z_{j,t} = 0$. Both properties are shown to hold for the replicator dynamics in Nachbar (1992, p.323) when the source of change in the population distribution is driven by the relative difference of profits from the population average under the restriction that the profit matrix T is non negative with strictly positive diagonal. In the case we are considering the matrix T is non negative but the diagonal is not strictly positive ($\pi_{1,1} = 0$); to keep forward invariance we then have to consider a dynamics based on the absolute difference of profits from the population average. Note infact that [FI] requires that each firm type is represented in the population throughout the evolution until the end when some of them will eventually die off; when the source of change is the absolute difference between profits, a sufficient condition for [FI] to hold is that $\overline{\Pi} < 1$ which is always the case in our setting[8].

4: Analysis of the equilibrium

We can establish some results analytically for the case of replicator dynamics with no noise. In order to do this, we consider a hypothetical "conjecture" game where there are two firms. The game has two stages, and in stage 1 the firms choose their conjectures ϕ, and in the second stage the outputs are determined given the chosen CVPs (according to [6]). We call this the conjecture game. It is easy to show that when c>0, there exists a unique Nash equilibrium in the conjecture game, and moreover the equilibrium is strict.

Formally, we can write the conjecture game as $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$. where the payoff function Π_i^* is given by [7]. We will show that when c>0, there exists a unique strict Nash equilibrium in conjectures. Note that here we are treating ϕ as a continuous variable, and not restricting it to Φ:

Proposition 1: Consider $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$. If c>0, then there exists a unique symmetric and strict Nash equilibrium ϕ^*. The conjecture ϕ^* is consistent, in that the slope of the decision rule equals the conjectured slope.

Proof.: see appendix.

The equilibrium conjectures in $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$ are the *consistent conjectures* of the constituent duopoly game. This is quite clear from the proof of Proposition 1: given the conjecture ϕ_b of the other firm b, the optimal conjecture of firm a is the "correct" conjecture: i.e. the CVP that corresponds to the actual slope of the other firm's reaction function in output space (given by [4b]). The unique Nash equilibrium occurs where both firms have correct conjectures about each other, which gives rise to the consistent conjectures equilibrium[9].

This result enables us to identify analytically the attractor of the noiseless replicator dynamics in the model. A crucial concept here is the notion of an evolutionary stable strategy (ESS).

[7]Gale, Binmore and Samuelson (1995) adopt a replicator dynamics similar to the one adopted here. In addition, they tell a specific story about the meaning of noise and its possible sources. For a different specification of noise giving rise to mutation see, among the others, Linster (1994).

[8]To see that consider the case of firm type ϕ=-1. This type of firm always earns zero profits and its proportion evolves according to $z_{-1,t+1} = z_{-1,t} + z_{-1,t}(-\overline{\Pi})$. It is clear that for $\overline{\Pi} < 1$, $z_{-1,t} = 0$ only as $t \to \infty$.

[9]This is not a special case due to quadratic costs. In general, equilibrium conjectures will be consistent, see Dixon, (1995) for a general proof.

Definition ESS: A strategy (firm type) ϕ is an ESS of the symmetric game $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$ if and only if:

$$\Pi(\phi,\phi) \geq \Pi(\phi',\phi)$$

and

$$\Pi(\phi,\phi) = \Pi(\phi',\phi) \Rightarrow \Pi(\phi,\phi') > \Pi(\phi',\phi') \textit{ for all } \phi' \neq \phi$$

where $\Pi(\phi_1,\phi_2)$ represents the payoff to a player playing strategy ϕ_1 against an opponent playing strategy ϕ_2.

An ESS strategy must be a Nash equilibrium. In addition, if it is a weak NE, the strategy needs to do strictly better against an alternate best reply then this latter does against itself. A sufficient condition for a pure strategy (firm type) to be an attractor of the replicator dynamics is that it is an Evolutionary Stable Strategy (ESS). A sufficient condition for a strategy to be ESS is that it is a *strict* Nash equilibrium (for a discussion of the relationship between Nash equilibrium strategies and ESS strategies see Linster (1990) and Robson (1992)). Thus we can see that in the case of strictly convex costs (c>0), and in the absence of noise the evolutionary replicator dynamics will give rise to consistent conjectures there being no other NE in the game.

In the case of c=0, however, things are rather different. Here the Nash equilibrium of $[\phi_i \in [-1,1], \Pi_i^*(\phi_i,\phi_j), i,j=1,2]$ is $\phi = -1$, which is the consistent conjecture with zero production costs. However, it is not a *strict* Nash equilibrium in the conjecture game $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$ since $\phi=-1$ is a dominated strategy. There is a unique Nash equilibrium with $\phi^* = -1$, (the consistent conjecture) which is a weak equilibrium and hence:

Proposition 2: Consider $[\phi_i \in [-1,1], \Pi_i^*, i=a,b]$. Let c=0. There exists no ESS.

Proof: See Appendix

So, whilst we do not need to simulate the noiseless replicator dynamics when c>0, we need to analyse the case of c=0 with more care, since there is no ESS. Whilst we abstracted from the

fact that the strategy space (the set of firm types) was finite in Propositions 1 and 2, this turns out to be crucial in the case of c=0. Since we are restricting firm types to be a finite set, it turns out that there does exist a set of Nash equilibria; with the exception of $\phi^* = -1$, these are in general strict. We will now proceed to analyse the Nash equilibria of the finite action version of this game.

It is easiest to analyse the discrete game where the CVPs are restricted to a grid of granularity δ by first considering the continuous version where the minimum deviation is δ. Let $\Pi(\phi_1,\phi_2)$ be the payoff from playing strategy ϕ_1 against strategy ϕ_2. For $\phi \in [-1,1]$, it is clear that

$$\frac{\delta\Pi(\phi_1,\phi_2)}{\delta\phi_1} = \begin{cases} >0 & \text{for } \phi_1 < \phi_1^* \\ <0 & \text{for } \phi_1 > \phi_1^* \end{cases} \quad \forall \phi_2 > -1 \tag{11}$$

where ϕ_1^* is the best response to ϕ_2. In words $\Pi(\phi_1,\phi_2)$ is strictly increasing to the left of the best response ϕ_1^* and strictly decreasing to the right of it.

Now consider the discrete case, where firm types are restricted to the grid, $\phi \in \Phi$. Property [11] implies that if

$$\Pi(\phi_i,\phi_j) < \Pi(\phi_j,\phi_j) \quad i<j \text{ then}$$

$$\Pi(\phi_k,\phi_j) < \Pi(\phi_j,\phi_j) \quad \forall k<i \tag{12}$$

moreover

$$\Pi(\phi_i,\phi_j) < \Pi(\phi_j,\phi_j) \quad \forall i>j \tag{13}$$

Equation [13] means that it never pays to play a strategy higher than that played by the opponent. This, in turn, allows us to concentrate only on lower strategies when checking for Nash equilibria. Equation [12] implies that if a strategy $\phi_i \in \Phi$ cannot be improved upon by its closest element to the left, this will be true for all other lower strategies. Define:

$$V(\phi_i,\delta) \equiv \Pi(\phi_i - \delta, \phi_i) - \Pi(\phi_i,\phi_i)$$

where $\delta>0$, $\phi_i \in \Phi$ and $\Pi(\phi_i, \phi_j)$ is the payoff to the firm playing strategy i against strategy j, and let

$$\hat{S}(\delta) \equiv \{\phi_i \in \Phi: V(\phi_i, \delta) \le 0\}$$

and let $\hat{\phi}(\delta)$ be the upper bound for $\hat{S}(\delta)$.

Clearly, as $\delta \to 0$, $V(\phi_i, \delta) \to 0$, $\hat{\phi}(\delta) \to -1$, $\hat{S}(\delta) \to \{-1\}$.

We can now state the first lemma which identifies the set of the Nash equilibria (NE) of the game:

Lemma 1

Let $V(\phi_i, \delta)$, $\hat{\phi}(\delta)$ and $\hat{S}(\delta)$ be defined as above, then:

a) ϕ_i is a NE iff $\phi_i \in \hat{S}(\delta)$;

b) if $V(\hat{\phi}(\delta), \delta) = 0$, then $\hat{\phi}(\delta)$ is a weak NE. If, on the contrary $V(\hat{\phi}(\delta), \delta) < 0$, then $\hat{\phi}(\delta)$ is a strict NE. All $\phi_i \in \hat{S}(\delta): \phi_i < \hat{\phi}(\delta)$ are strict NE.

Proof: See Appendix.

Before stating the main result, we rule out the possibility of having an attractor of the replicator dynamics involving $1 < N \le n$ surviving firm types, such that some of them lies outside $\hat{S}(\delta)$.

Let $Z^T \equiv \{z_1, z_2, ..., z_n\}$ be a generic state vector representing the distribution of the population across firm types at time T such that $z_i \ge 0$ $i = 1, 2, .., n$, $\sum_i z_i = 1$ and let $1 < N \le n$ be the number of firm types to which Z^T gives positive support. Then define

$$\sigma \equiv \{\phi_1, \phi_2, ..., \phi_N\}$$

as the ordered set of firm types surviving with $\phi_N \notin \hat{S}(\delta)$.

Lemma 2

Z^T is not an attractor of the replicator dynamics.

Proof: See the Appendix.

We can now state the main result.

Proposition 3

Let $\delta > 0$ and $\phi_i^* > -1$, then ϕ_i^* is an attractor for the replicator dynamics iff $\phi_i^* \in \hat{S}(\delta)$.

Proof: See the Appendix.

5: The results of the simulations.

A: Evolution without noise.

As mentioned beforehand, the sample considered consisted of 101 firm types with CVPs ranging from -1 to 1. For this population $\hat{\phi}(0.02) = -0.9$ and $\hat{S}(0.02) = \{\phi_i : \phi_i = -1 + 0.02 * i, \quad i = 0..5\}$. The simulations ran until the difference of surviving firms profits from the population average where smaller than 1e-100, i.e. $\Pi_i - \overline{\Pi} \le 1e-100$. The first simulation was run with no noise and with the initial vector $Z_i^0 = 1/101 \quad \forall i = 1,.., 101$. In this case there is one surviving firm. This has a CVP equal to -0.9 which corresponds to $\hat{\phi}(\delta)$. The result clearly indicates a lack of cooperation.The equilibrium average payoff equal to 0.0226 is only the 18.1% of the payoff deriving from joint profit maximisation. The surviving firm performed pretty badly in the Tournament, its rank being 96, where about a half of the firms expected the rival to be cooperative (CVP>0) and the other half expected it to be competitive (CVP<0).

It is worth to recall that the decision rule given above by [4a,b] implies that when faced with the prospect of being matched with a *cooperative* rival the firms behave *cooperatively*; in terms of fig.1(a), the reaction function becomes steeper as the CVP moves towards 1[10]. Consequently, being cooperative proves to be good only against other cooperative firms. On the contrary, being competitive, i.e. expecting the rival to behave competitively, proves to be good in both events. If the rival firm actually behaves competitively it is better to be competitive, and if the rival is cooperative defection is more rewarding than cooperation. For these reasons

[10] In our context a cooperative outcome like the joint profit maximisation is reached when both firms have a CVP equal to one. In all symmetric equilibria where both firms have a positive CVP industry profits are higher the closer the CVP is to unity and lower the more negative the CVP is. For these reasons we deem as cooperative (competitive) those firms with positive (negative) CVP.

cooperative firms tend to be eliminated once the evolution begins; as the proportion of such firms declines, the average payoff of the remaining ones declines too, being smaller and smaller the probability of being matched with another *cooperative* firm.

On the *competitive* side, the Walrasian firm always earns zero profits and rapidly disappears; very aggressive firms (those with CVPs close to -1) are not very successful when they meet each other and when confronted with cooperative firms. As the evolution proceeds firms like the Stackelberg leader (SLF) tend to prosper, which in turn induces a reduction in the overall average profit as the presence of cooperative firms reduces; as the degree of cooperation goes down, however, the more competitive firms (those with a CVP closer to -1) find themselves in an environment more suitable to their characteristics. (the idea is that against cooperative firms, less aggressive firms do better than those with CVP closer to -1).

Fig.3 here

Fig.3 shows the evolution of four firm types with a CVP ranging from -0.84 to -0.9 in five millions iterations. In the early stages of the evolution (roughly 25000 iterations) firm type -0.84 prospers. Subsequently it is replaced by firm type -0.86 which dominates with a proportion almost equal to one for roughly 50000 iterations. The same kind of process takes place as the evolution unfolds until firm type -0.9 establishes itself as the modal firm. From this point onwards (for almost 2.5 millions iterations) we do not observe cycles and the population eventually reaches its equilibrium where the entire population of firm types plays the same strategy. This conclusion is in line with the predictions of Proposition 3 which states that $(\hat{\phi}(\delta),\hat{\phi}(\delta))$ is a potential attractor for the replicator dynamics.

To test the robustness of this result we run 10 evolutions with random initial proportion vectors. In all cases the surviving firm was that with CVP=-0.9. This seems to indicate that although $(\hat{\phi}(\delta),\hat{\phi}(\delta))$ is not a global attractor for the replicator dynamics, its basin of attraction is large. To explore this issue further we constructed 27 initial distributions with modal firms

ranging from -0.8 to 0.8 and with different levels of variance (high, medium and low respectively). These distributions cover a wide range of cases, as they take very skewed distribution and systematically favour certain groups of firms, the initial distribution of firms is depicted in fig. 4

fig.4 here

The results obtained are unanimous; the surviving firm type being -0.9 in all the simulations. A word of caution should be mentioned here because of the problem of "computer underflow" as identified by Nachbar (1992). When running simulations of this kind, it is necessary to specify some cut-off: if variables (Z_i or Π_i) fall below a certain cut-off, it is set to zero[11]. Therefore, some firm types can become extinct before being able to invade a population whose distribution across firm types has become suitable for them to prosper. That this is quite possible can be seen from fig.3 which show that in a "typical" simulation firm types other than the final attractor can achieve population proportion very close to unity. Hence it is quite possible to obtain "pseudo-convergence", where the simulation converges prematurely due to the extinction generated by the cut-off rule.

We are fully aware that this arbitrary extinction rule exploits the less appealing of the properties of the replicator dynamics, namely the "no creation" property but we will argue that, for our purposes, this problem is not as serious as it may seem. The results of the simulations have been, infact, almost fully predicted. From Proposition 3 we know that the only candidates to the equilibrium lie in $\hat{S}(\delta)$. Our interest rests in the determination of which of the firm types contained in $\hat{S}(\delta)$ will be selected as attractor of the replicator dynamics when different initial distributions are considered; as long as the problem of computer underflow does not

[11]We chose a cut-off rule of 1e-200 because of the time constraint; the package we used can infact handle number as small as $4.19*10^{-307}$. With no such cut-off rule the whole process of the simulation become extremely time consuming.

substantially alter the dynamics of the system in a way that cannot be predicted affecting the possibility for these firm types to become attractors of the dynamics, this will not be a problem. In our simulations we have expressly checked for the problem of pseudo-convergence created by "computer underflow" and their results should be considered immune from it.

Although the firm type ϕ=-0.9 it is not a global attractor for the replicator dynamics, the outcome generated by these simulations clearly indicates that there is a strong tendency towards competition. The level of profit resulting from the evolutions is indeed always well below the collusive level of profits. In addition to that we generated another 18 initial distributions using a different deterministic algorithm. As in Dixon *et al.* (1994), starting from the first point in the grid, we picked the first 10 firm types and gave to them a weight 1/k (total 10/k) and the rest of the firms an equal part of the remaining weight ((k-10)/(91*k)). The next firm is missed out, and then the next ten are picked and the new initial proportions are calculated as before and so on. The first nine distributions were computed with k=40 (which gives the 10 firms group a proportion of 0.025 each), the other 9 with k=20 (which gives the 10 firm group a proportion of 0.05 each). In this way firm types all over the grid were given a chance to start off with a large initial proportion.

Again firm type -0.9 resulted the attractor in all of them. It is interesting to note that none of the firm types with CVP<-0.9 become attractors even when the initial distribution contains a large proportion of competitive firms suggesting that their basin of attraction is very small. To check this issue further we, lastly, run a simulation with an initial distribution of firms across types such that the first 10 types were given z_i=0.09 for i=1,..,10 and the remaining ones z_j=(0.01/91). In this case too, firm type -0.9 resulted to be the attractor of the dynamics.

B: Evolution with noise.

We ran several simulations for different levels of noise and for different values of the cost parameter c. When noise is introduced, a fraction d of firms undergo a random mutation which, according to the particular specification adopted, does not introduce new firm types. The mutant

firms switch to any other firm type with equal probability (1/101) and, consequently, all firms survive with at least a share of (d/n). For convergence we adopted the following criterion:

$$z_i^{t+1} - z_i^t \leq 1e-100,$$

in other words, the simulations ran until the firms proportions were almost constant.

Table 5 here

In table 5 the first column gives the level of noise; the second column the conjecture of the modal firm type; the third the proportion of the firms which are of the modal type; and lastly the mean firm type (the weighted average conjecture, with population proportions as weights). For very low levels of noise the modal firm is still -0.9 as in the noiseless case. Furthermore, the proportion of the modal firm remains high, as is reflected in the mean firm being only slightly less than -0.9. However, the presence of noise means that all firm types survive with at least proportions d/101. Whilst less competitive firms die out and their population share goes to zero in the noiseless replicator dynamics, with noise theses firms survive and make the environment less competitive (more cooperative). For levels of noise at or above d=0.001, we find that the mean and modal firms are becoming significantly less competitive as a result, and that the proportion of the modal firm declines. This of course reflects the fact that the reaction function for conjectures is upward sloping, so that less competitive opponents favour less competitive firms.

Table 6 here.

In Table 6 we have simulations of the noisy replicator dynamics for different values of the cost parameter (c=0.1, 0.3, 0.5 and 1), and four different levels of noise. As the first three lines relate to simulations with c=0.1: line one gives the modal firm, line 2 its proportion, line three the mean firm. The next three lines relate to c=0.3 and so on. The four columns represent different levels of noise for each case. The results are very much in line with what we found for the case of c=0. For low level of noise, the mean and modal firm are almost the same (representing the fact that the share of the modal firm is very high), and the modal firm is the consistent conjecture. However, as the level of noise increases, the mean/mode become different and both become less competitive, for the same reasons as were apparent in the no cost case. Note that the consistent conjecture becomes less competitive as c increases (this is a standard result - see Bresnahan (1981) and Dixon (1986)).

Conclusion.

In this paper, we have taken a model in which firm behaviour depends on firm beliefs. Firms play the conjectural variation duopoly, and their belief is their "conjecture" about the slope of other firms reaction functions. We analyze this in the context of an evolutionary framework, in which more successful types of rule become more common. We model the Darwinian process using replicator dynamics. Analytically, we find that in the case of strictly convex costs, the unique ESS is the consistent conjecture. Evolution will select firms with beliefs that are correct. In the case of no costs, there exists no ESS, and the consistent conjecture (Bertrand) is a dominated type. We consider the case of a finite susbset of possible conjectures, and use these to simulate the replicator dynamics. Analytically, we find that there is a set of possible attractors contained within an compact convex interval with lower bound at the consistent conjecture. The Pareto Dominant conjecture that is an attractor is the least competitive. We tried many simulations, and in all of them this Pareto Dominant attractor was selected by the replicator dynamics. Whilst there are other attractors, their basin of attraction appears to be very small. We also consider the case of noisy replicator dynamics using simulations, for both the case of strictly convex costs and zero costs. In both cases we found the same broad results. At low levels of noise, the noiseless attractor (the Pareto dominant attractor) was the modal firm, with a population share very close to unity. However, as the noise increases, the mean and modal firm

both become less competitive, moving away from the consistent conjecture. We believe that the framework of this paper provides a rationale for consistency of conjectures which is based on bounded rationality and evolutionary selection. Most of the criticisms of the consistency and conjectural variations have focussed on the issue from the perspective of classical game theory based on perfect rationality and common knowledge. Whilst we accept these criticisms as valid within the their own framework, our approach is based on different and we believe more plausible foundations.

Appendix

Proof of Proposition 1

Maximisation of the payoffs functions as given in [7] with respect to ϕ_i i=1,2, yields the best response function:

$$\phi_i^*(\phi_j) = -\frac{1}{2+c+\phi_j} \qquad i, j = 1,2\ i \neq j$$

which corresponds to the actual slope of the rival's reaction function in output space, [4]. Solving the above system yields the Nash equilibrium in conjectures:

$$\phi_i^* = \phi_j^* = -\frac{1}{2}c - 1 + \sqrt{\frac{4c+c^2}{4}} \qquad \text{[A1]}$$

Uniqueness of the equilibrium can easily be established noting that $\phi_i^*(\phi_j)$ is a contraction mapping. Its first derivative is

$$0 < \frac{d\phi_i^*(\phi_j)}{d\phi_j} = \frac{1}{(2+c+\phi_j)^2} < 1$$

since $\phi \in [-1, 1]$ and since $c \geq 0$, this implies that the solution [A1] is unique.

QED

Proof of Proposition 2

The first part of the proposition is similar to that of proposition 1 and therefore is only sketched. The optimal conjectures pair derived from maximisation of profits w.r.t. ϕ is:

$$\phi_i^*(\phi_j) = -\frac{1}{2+\phi_j} \qquad i, j = 1,2\ i \neq j$$

the equilibrium in conjectures is $\phi_i^* = \phi_j^* = -1$. Again uniqueness can be proved noting that $\phi_i^*(\phi_j)$ is a contraction mapping.

We then show that $\phi_i^* = \phi_j^* = -1$ is not ESS. From definition of ESS, it must be that

$$\Pi(-1,-1) \geq \Pi(\phi,-1) \ \textit{and if} \ \Pi(-1,-1) = \Pi(\phi,-1) \ \textit{then} \ \Pi(-1,\phi) > \Pi(\phi,\phi)$$

noting that $\Pi(-1,\phi) = \Pi(\phi,-1) = 0 \ \forall\phi$, and that $\Pi(\phi,\phi) > 0 \ \forall\phi \neq -1$, the only Nash equilibrium of the game, given by $\phi_i^* = \phi_j^* = -1$ is not ESS. Moreover being $\phi_i^* = \phi_j^* = -1$ unique, there are no others ESS.

QED

Proof of Lemma 1

A) We first prove necessity. $V(\phi_i,\delta)$ is increasing and concave in ϕ, for all $\phi_i > \hat{\phi}(\delta)$, $V(\phi,\delta) > 0$ and hence by [13] each firm has an incentive to deviate from (ϕ_i,ϕ_i) by reducing its strategy to $(\phi_i - \delta)$; $\phi_i > \hat{\phi}(\delta)$ is not a best response against itself and hence cannot qualify for a NE. Sufficiency is easily proved by contradiction. Suppose on the contrary that $\tilde{\phi} \in \hat{S}(\delta)$ is not a NE, from [13] this will imply that

$$\Pi(\tilde{\phi}-\delta, \tilde{\phi}) \geq \Pi(\tilde{\phi}, \tilde{\phi})$$

which immediately contradicts the definition of $\hat{S}(\delta)$.

B) In $\hat{\phi}(\delta)$ we have that

$$\Pi(\hat{\phi}(\delta)-\delta, \hat{\phi}(\delta)) \leq \Pi(\hat{\phi}(\delta), \hat{\phi}(\delta)) \qquad \text{[A2]}$$

and hence by [12]

$$\Pi(\phi', \hat{\phi}(\delta)) < \Pi(\hat{\phi}(\delta), \hat{\phi}(\delta)) \quad \forall\phi' \leq \hat{\phi}(\delta) - \delta \qquad \text{[A3]}$$

If condition [A2] holds as an equality, in $[\hat{\phi}(\delta), \hat{\phi}(\delta)]$ and $[\hat{\phi}(\delta)-\delta, \hat{\phi}(\delta)]$ there is no incentive to deviate from one's own strategy and thus they represent two weak NE, when, instead it does hold as a strict inequality $[\hat{\phi}(\delta), \hat{\phi}(\delta)]$ is strict NE. Concavity of $V(\phi_i,\delta)$ in ϕ ensures that $\phi_i \in \hat{S}(\delta)$: $\phi_i < \hat{\phi}(\delta)$ *are all strict NE.*

QED

Proof of Lemma 2.

Assume on the contrary that Z^T is an attractor, this would imply that in T $AP_i^T = AP_j^T \quad \forall i, j \in \sigma$ where AP_i^T is the average profit for firm type i at time T. Consider the firm type $\phi_N - \delta$, if $(\phi_N - \delta) \in \sigma$ then in T

$$AP_{(\phi_N)} = AP_{(\phi_N - \delta)} \qquad [A4]$$

if $(\phi_N - \delta) \notin \sigma$ then, by forward invariance (see Nachbar, 1992), for some τ<T Z^τ supports $\phi_N - \delta$. If the system has to move from Z^τ to Z^T then by the boundary property (Nachbar, 1992) we should have that for $\tau < t \leq T$:

$$AP_{(\phi_N)} > AP_{(\phi_N - \delta)} \qquad [A5]$$

The average profits are given by

$$AP_{(\phi_N)} = \sum_{j=1}^{N} z_j \Pi(\phi_N, \phi_j) \text{ and}$$

$$AP_{(\phi_{N-\delta})} = \sum_{j=1}^{N} z_j \Pi(\phi_{N-\delta}, \phi_j).$$

Recall that by [11] $\forall \phi_j \; \exists \phi_i^*(\phi_j)$, where $\phi_i^*(.)$ denotes the best response to ϕ_j, and that as ϕ_i^* gets closer to $\phi_i^*(.)$ the profits are increasing. Since $\phi_N \notin \hat{S}(\delta)$, from Lemma 1 we know that it is not a best response to itself i.e.

$$\Pi(\phi_N - \delta, \phi_N) > \Pi(\phi_N, \phi_N),$$

moreover by [13]

$$\Pi(\phi_N - \delta, \phi_j) > \Pi(\phi_N, \phi_j) \quad \forall \phi_j \leq \phi_N - \delta \in \sigma$$

it follows that

$$AP_{(\phi_N)} < AP_{(\phi_N - \delta)} \quad \forall Z^t$$

which contradicts both [A4] and [A5].

QED

Proof of Proposition 3.

We first prove that $\phi^* \in \hat{S}(\delta)$ is a sufficient condition. All $\phi \in \hat{S}(\delta)$ are strict NE and hence are evolutionary stable strategies (ESS). ESS is, in turn, a sufficient condition for ϕ being an attractor of the replicator dynamics.

We then show necessity. Suppose the contrary, i.e. ϕ^* is an attractor but $\phi^* \notin \hat{S}(\delta)$. Let Z^T be the (degenerate) equilibrium distribution of the population across firm types. By forward invariance, close to the limit, i.e. for some τ<T, let Z^τ define the state vector

$$Z^\tau \equiv \{z_1^\tau, z_2^\tau, .., z_M^\tau\}$$

where $z_i \geq 0 \; i = 1, 2, .., M, \; \sum_i z_i = 1$, and let ϕ^* be the k^th^ surviving firm typewhere $1 \leq k \leq M$, and z_k^τ be the proportion of firm type ϕ^*. The average profit of firm type i at time τ is

$$AP_i = \sum_{j \neq k} z_j^\tau \Pi(\phi_i, \phi_j) + z_k^\tau \Pi(\phi_i, \phi^*) \qquad [A6]$$

Arbitrarily close to the limit all $z_j^\tau \quad j \neq k$ are negligible and the average profit of each firm type can be compared considering the second term of the RHS of [A6] only. The transition from Z^τ to Z^T requires that

$$\Pi(\phi^*, \phi^*) > \Pi(\phi_i, \phi^*) \quad \forall i \neq k \qquad [A7]$$

Expression [A7] is always true for i>k given [13]

$$\Pi(\phi^*, \phi^*) > \Pi(\phi_i, \phi^*) \quad \forall i > k,$$

moreover, given [11], we are sure that if

$$\Pi(\phi^*, \phi^*) > \Pi(\phi_j, \phi^*) \quad \forall j < k$$

then

$$\Pi(\phi^*, \phi^*) > \Pi(\phi_i, \phi^*) \quad \forall i < j$$

Therefore we can restrict our attention to the comparison between $\Pi(\phi^*,\phi^*)$ and $\Pi(\phi^*-\delta,\phi^*)$ only. Condition [A7] then becomes

$$\Pi(\phi^*,\phi^*) > \Pi(\phi^*-\delta,\phi^*)$$

which never holds when $\phi^* \notin \hat{S}(\delta)$.

QED

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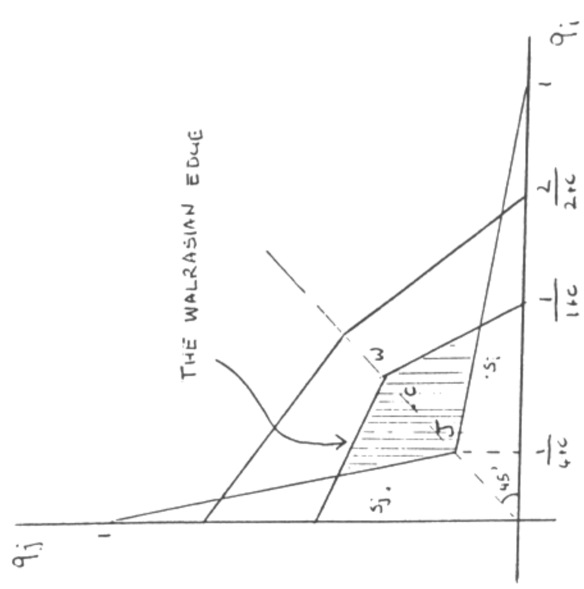


FIG. 1b

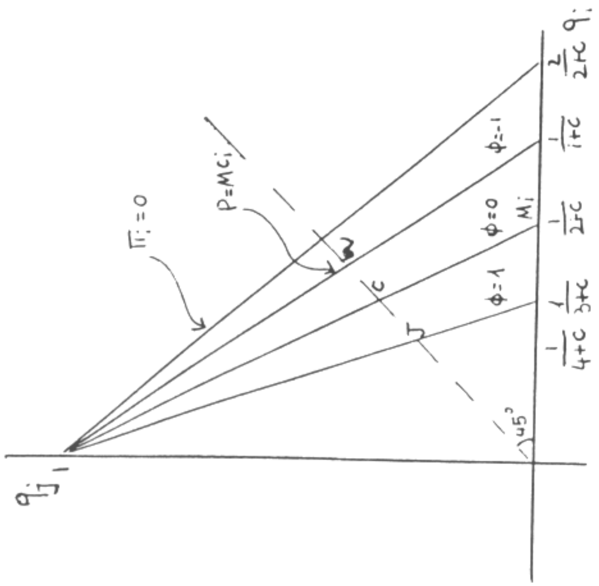


FIG. 1a

Fig.2 The average tournament profits for different values of c

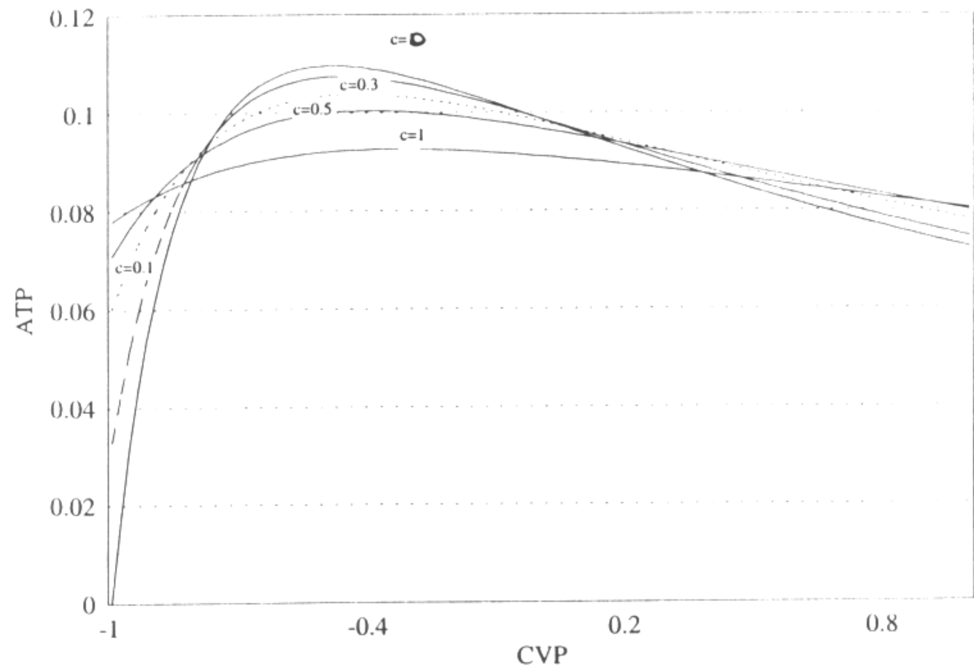


Fig.3 Evolution of the distribution of firms across firm types

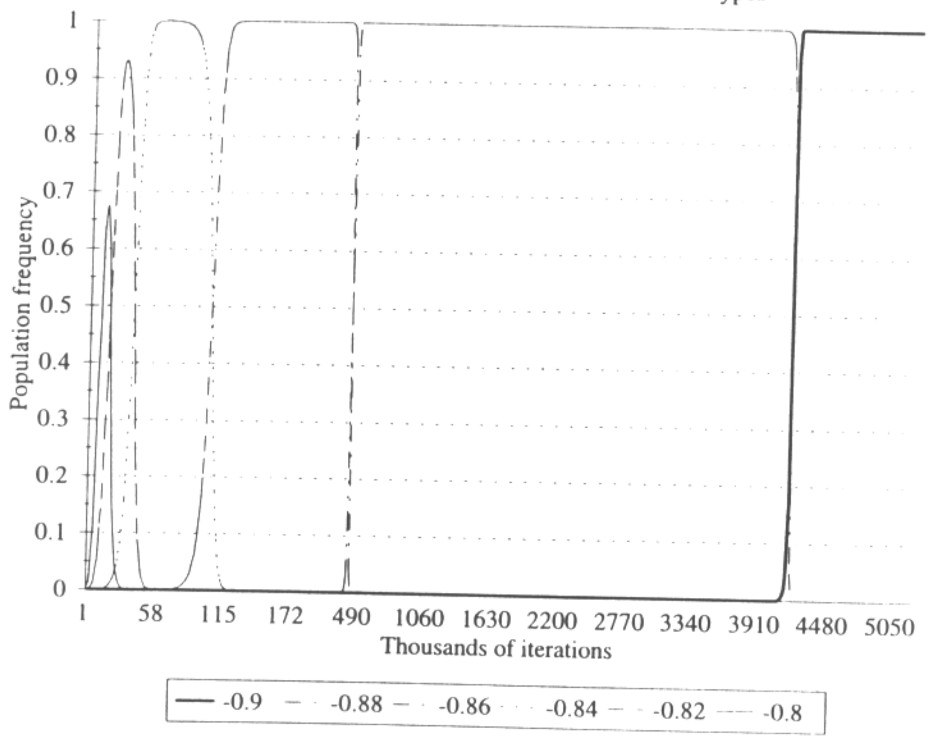
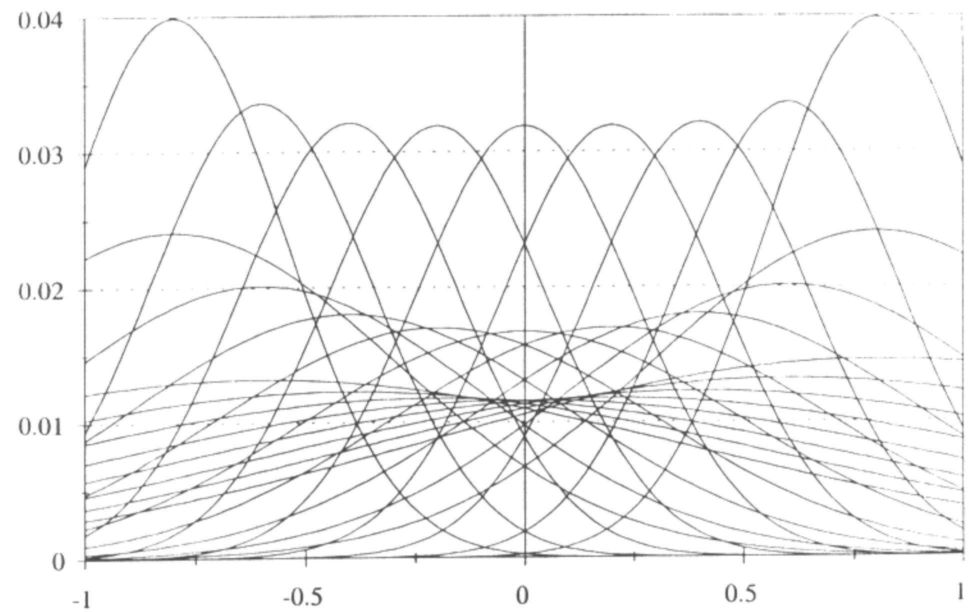


Fig.4 27 Initial Firms' Distributions



	modal firm	z	Mean firm
d=1e-5	-0.9	0.996	-0.899
d=0.0001	-0.9	0.976	-0.894
d=0.001	-0.84	0.71	-0.792
d=0.01	-0.62	0.047	-0.359
d=0.05	-0.52	0.014	-0.067
d=0.1	-0.5	0.012	-0.025

Table 5: The evolutionary results for different levels of noise with c=0

		d=1e-5	d=0.0001	d=0.001	d=0.01
	modal firm	-0.72	-0.72	-0.70	-0.56
c=0.1	z	0.99	0.94	0.40	0.04
ϕ^*=-0.72	mean firm	-0.72	-0.72	-0.65	-0.30
	modal firm	-0.58	-0.58	-0.56	-0.48
c=0.3	z	0.99	0.87	0.23	0.03
ϕ^*=-0.58	mean firm	-0.58	-0.58	-0.52	-0.21
	modal firm	-0.50	-0.50	-0.48	-0.44
c=0.5	z	0.98	0.79	0.15	0.02
ϕ^*=-0.5	mean firm	-0.50	-0.49	-0.43	-0.16
	modal firm	-0.38	-0.38	-0.38	-0.34
c=1	z	0.95	0.59	0.07	0.02
ϕ^*=-0.38	mean firm	-0.38	-0.37	-0.30	-0.08

Table 6: The evolutionary results for different levels of noise with c>0.